Influence of electron collisions on the resonance cone phenomenon in a cold magnetized plasma

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We present a theoretical study of the potential produced by an antenna immersed in a cold magnetized plasma. This phenomenon is described in the literature as the ''resonance cone'' phenomenon. In this work, we take into account electron collisions with other particles (neutral or charged). We show that the domain—in terms of frequencies—where the resonance cone exists is drastically reduced for a collisional plasma. Furthermore, the resonance cone peak is shifted by collisions, so that the usual formula used to compute the electronic density is not quite exact. All the calculations are done with a finite magnetic field. [S1063-651X(96)50508-8]

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The resonance cone phenomenon (on the upper hybrid branch) has been extensively studied during the past thirty years, and especially since the first experiment of Fisher and Gould, who showed it could be used to measure the electronic density of a magnetized plasma [1–3]. Gonfalone used this method as a plasma diagnostic, measuring the electronic density and the electronic temperature [4]. Neglecting collisions, the electronic density is known by measuring only the resonance cone angle. The electronic temperature is measured using the interference structure which appears around the resonance cone, via the measurement of the angle between two successive maxima of the structure. This structure has been explained as a consequence of the thermal effects [3]. These studies are making reference to what is called the ''cold cone,'' that is to say without thermal effects: actually thermal effects shift very lightly the resonance cone peak. In other respects, a cold plasma could be highly collisional with respect to the other typical plasma frequencies (e.g., magnetic field $B=50$ G, electron density $n_e=10^{11}$ cm^{-3}, electronic temperature $T_e=0.1$ eV: electron-ion collision frequency $\nu_{ei}\approx 0.1\omega_{ce}$, plasma electronic pulsation $\omega_{pe}\approx 20\omega_{ce}$ with ω_{ce} the electronic cyclotronic pulsation; in addition, we must take into account the electron collisions on neutral particles). The influence of collisional effects on the cold cone has not been studied so far, to our knowledge. In this work, we use a simple analytical collisional model to study the influence of electron collisions on the resonance cone.

A typical experimental setup is described in Fig. 1: an antenna immersed in a cold magnetized plasma emits a wave. The magnetic field is uniform and externally produced.

The signal, received by another antenna moving around the first one, could be considerably enhanced on the resonance cone, when it exists. If we do not take into account the thermal velocity in the dispersion relation and if we neglect collisions of electrons with other particles, the equation of the resonance cone is $z^2+\rho^2(K_\parallel/K_\perp)=0$ where z is the coordinate along the magnetic field and ρ the polar coordinate, with $K_\perp$ and $K_\parallel$ the diagonal terms of the reduced dielectric tensor: $K_\perp=1-[\omega_{pe}^2/(\omega^2-\omega_{ce}^2)]$ and $K_\parallel=1-(\omega_{pe}^2/\omega^2)$, with ω the frequency of the wave.

The equation $z^2+\rho^2(K_\parallel/K_\perp)=0$ leads to the usual formula, used to determine the electron density, knowing the magnetic field and the wave frequency:

$$\sin^2\kappa=\frac{\omega^2(\omega_{pe}^2+\omega_{ce}^2-\omega^2)}{\omega_{pe}^2\omega_{ce}^2}), \tag{1}$$

where κ is the cone angle (Fig. 1) [1–4].

Therefore the condition of existence of the resonance cone is $K_\parallel K_\perp<0$, which gives

$$\omega<\min(\omega_{ce},\omega_{pe}) \quad \text{(lower branch)}$$

$$\max(\omega_{ce},\omega_{pe})<\omega<\omega_{\rm UH} \quad \text{(upper hybrid branch)},$$

where $\omega_{\rm UH}$ is the upper hybrid frequency: $\omega_{\rm UH}=\sqrt{\omega_{pe}^2+\omega_{ce}^2}$. When we take into account electron collisions with ions or with neutral particles, these conditions are no longer valid. We need to recalculate the potential with the collision frequency ν.

Near the emitting antenna ($r\ll 2\pi c/\omega$, where r is the distance from the antenna) and near the resonance cone peak (at resonance the wave number $k\to\infty$) the electrostatic potential can be computed using the quasistatic equation: $\vec{\nabla}\cdot\vec{D}=\rho_{\rm ext}$, with $\vec{D}$ the electric induction and $\rho_{\rm ext}$ the charge density created by the emitting antenna as an electric oscillating point at frequency ω, localized at the origin, and given by [3] $\rho_{\rm ext}=q_e\exp(-i\omega t)\delta(\vec{r})$, with q_e the elementary electric charge, i the complex number (0,1), t the time, δ the Dirac function, and $\vec{r}$ the position vector (Fig. 1).

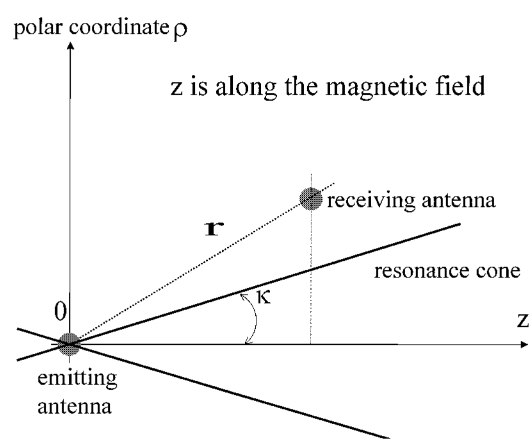

FIG. 1. Principle of the experiment.

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For an anisotropic medium, the relationship between the electric induction and the electric field $\vec{E}$ is $\vec{D}=\varepsilon_0\bar{\bar{K}}\cdot\vec{E}$, where $\varepsilon_0\bar{\bar{K}}$ is the dielectric tensor, with:

$$\bar{\bar{K}}=\left(\bar{\bar{1}}+i\frac{\bar{\bar{\sigma_e}}}{\omega\varepsilon_0}\right), \tag{2}$$

where $\bar{\bar{\sigma_e}}$ is the conductivity tensor. The tensor $\bar{\bar{K}}$ can be written as

$$\bar{\bar{K}}=\begin{bmatrix} K_\perp & -iK_\times & 0 \\ iK_\times & K_\perp & 0 \\ 0 & 0 & K_\parallel \end{bmatrix}, \tag{3}$$

for a finite magnetic field. The diagonal terms, taking into account electron collisions with the other particles (ions and neutral particles) via the electron collision frequency ν, are [5]

$$K_\perp=1-\frac{\omega_{pe}^2(\omega+i\nu)}{\omega[(\omega+i\nu)^2-\omega_{ce}^2]}, \quad K_\parallel=1-\frac{\omega_{pe}^2}{\omega(\omega+i\nu)},$$

neglecting ionic terms. The electron collision frequency ν represents losses for the wave, propagating in the plasma. By its presence, the dielectric tensor becomes complex. Then, as part of the quasistatic approximation, since $\vec{E}=-\vec{\nabla}\cdot\varphi(\vec{r},t)$, we have

$$\vec{\nabla}\cdot(\bar{\bar{K}}\cdot\vec{\nabla}\cdot\varphi(\vec{r},t))=-\frac{q_e}{\varepsilon_0}\exp(-i\omega t)\,\delta(\vec{r}). \tag{4}$$

Taking the Fourier transform of this equation, we obtain

$$\hat{\varphi}(\vec{k},t)=\frac{q_e}{\varepsilon_0}\frac{\exp(-i\omega t)}{(2\pi)^{3/2}}\frac{1}{(k_\perp^2K_\perp+k_\parallel^2K_\parallel)}, \tag{5}$$

where $k_\parallel=\vec{k}\cdot\vec{e}_z$ and $k_\perp=\vec{k}\cdot\vec{e}_x/\cos\psi=\vec{k}\cdot\vec{e}_y/\sin\psi$ with $\vec{k}$ the wave vector and $\hat{\varphi}$ the Fourier transform of the potential. Figure 2 describes the axis coordinates of the integration domain.

Taking cylindrical coordinates for the wave vector $d^3k\equiv k_\perp dk_\perp dk_\parallel d\psi$, we obtain by inverse Fourier transform

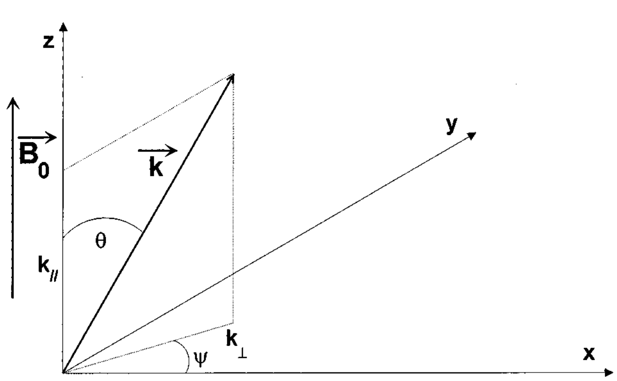

FIG. 2. Axis coordinates of the integration domain.

$$\varphi(\vec{r},t)=\frac{q_e}{\varepsilon_0}\frac{\exp(-i\omega t)}{K_\perp(2\pi)^3}\frac{1}{2}\int\int_{(\wp)}\int\frac{\exp(i\vec{k}\cdot\vec{r})k_\perp dk_\perp dk_\parallel d\psi}{k_\perp^2+k_\parallel^2\frac{K_\parallel}{K_\perp}}, \tag{6}$$

where the integration domain $(\wp)$ is defined as

$$(\wp)=\{(k_\perp,k_\parallel,\psi)\in\mathbb{R}^3 \text{ with } k_\perp\in\mathbb{R}^+;k_\parallel\in\mathbb{R};\psi\in[-\pi,\pi]\}. \tag{7}$$

Then the first integration, along ψ, gives, taking cylindrical space coordinates (ρ,z)

$$\varphi(\rho,z,t)=\frac{q_e}{\varepsilon_0}\frac{\exp(-i\omega t)}{K_\perp(2\pi)^2}\times\frac{1}{4}\int_{-\infty}^{+\infty}\left\{\int_{-\infty}^{+\infty}\frac{k_\perp\mathcal{H}_0^{(1)}(k_\perp|\rho|)dk_\perp}{k_\perp^2+k_\parallel^2\frac{K_\parallel}{K_\perp}}\right\}\times\exp(ik_\parallel z)dk_\parallel \tag{8}$$

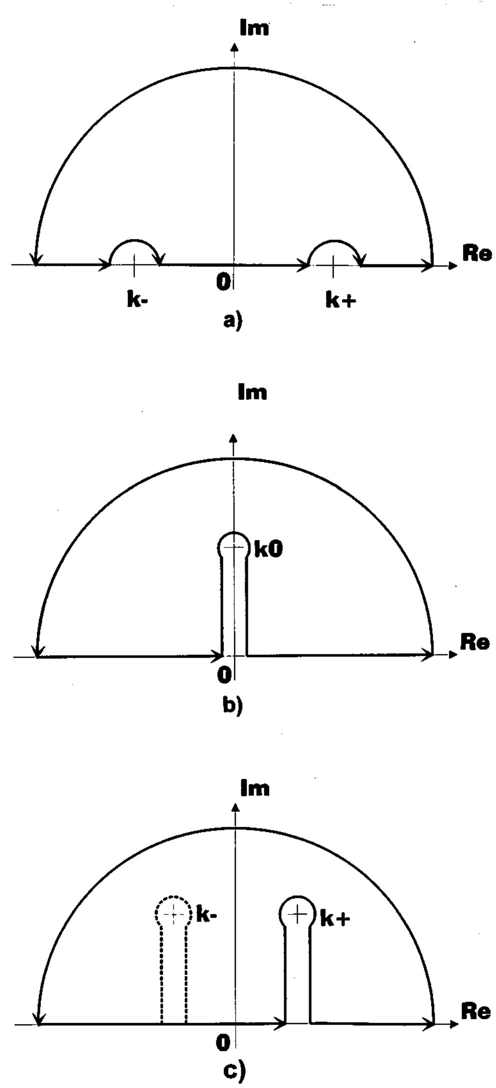

FIG. 3. Integration path. (a) First case: two half contributions of two poles; (b) second case: one pole contribution; (c) third case: one pole contribution, the one with the positive real part.

where $\mathcal{H}_0^{(1)}$ represents the Bessel function of the third kind (or Hankel function) of zeroth order. This calculation has been made by Kuehl [6,7]. For the two following integrations, we present a new calculation, taking into account electron collisions. Respecting the Jordan lemma [8], the integral

$$\mathcal{I}(k_\parallel,\rho)=\int_{-\infty}^{+\infty}\frac{k_\perp\mathcal{H}_0^{(1)}(k_\perp|\rho|)dk_\perp}{k_\perp^2+k_\parallel^2\dfrac{K_\parallel}{K_\perp}} \tag{9}$$

converges. For the second integration, depending on $K_\parallel/K_\perp$, we have three different cases depending on the integration path.

(a) $\mathrm{Im}[(-K_\parallel/K_\perp)^{1/2}]=0$ (half contributions of two poles k_- and k_+): the case of collisionless plasma, when a resonance cone exists, that is to say when $K_\parallel/K_\perp<0$ [cf. Fig. 3(a)]. The calculation of the integral $\mathcal{I}(k_\parallel,\rho)$ gives $-\pi\mathcal{N}_0(k_+|\rho|)$, where $\mathcal{N}_0$ represents the Bessel function of the second kind (or Neumann function), of zeroth order and with $k_+=|k_\parallel|(-K_\parallel/K_\perp)^{1/2}$ and $k_-=-k_+$.

The third integration gives [9]

$$\varphi(\rho,z,t)=\frac{q_e}{8\pi\varepsilon_0}\frac{\exp(-i\omega t)}{K_\perp}\frac{1}{\left(z^2+\rho^2\dfrac{K_\parallel}{K_\perp}\right)^{1/2}}. \tag{10}$$

(b) $\mathrm{Im}[(-K_\parallel/K_\perp)^{1/2}]>0$ and $\mathrm{Re}[(-K_\parallel/K_\perp)^{1/2}]=0$ (one pole k_0): the case of collisionless plasma, nonexistence of a resonance cone, that is to say when $K_\parallel/K_\perp>0$ [cf. Fig. 3(b)]. The calculation of the integral $\mathcal{I}(k_\parallel,\rho)$ gives $i\pi\mathcal{H}_0^{(1)}(k_0|\rho|)$, with $k_0=i|k_\parallel|(K_\parallel/K_\perp)^{1/2}$. Then the third integration gives the same result as in (a) [see Eq. (10)].

(c) $\mathrm{Im}[(-K_\parallel/K_\perp)^{1/2}]>0$ and $\mathrm{Re}[(-K_\parallel/K_\perp)^{1/2}]\neq 0$ (one pole $k_\pm=k_-$ or k_+ depending on which pole has a positive imaginary part): the case of collisional plasma, that is to say when $\nu\neq 0$ [cf. Fig. 3(c)]. The calculation of the integral $\mathcal{I}(k_\parallel,\rho)$ gives $i\pi\mathcal{H}_0^{(1)}(k_\pm|\rho|)$, with $k_\pm=\pm|k_\parallel|(K_\parallel/K_\perp)^{1/2}$. Then the third integration gives the same result as in (a) and (b) [see Eq. (10)].

So in every case, we find at last the *same result* for the potential $\varphi(\rho,z,t)$ [see Eq. (10)].

Thus, the potential modulus is

$$|\varphi(\rho,z,t)|=\frac{|q_e|}{8\pi\varepsilon_0}\frac{1}{|K_\perp|}\frac{1}{[(z^2+\rho^2a_\nu)^2+(\rho^2b_\nu)^2]^{1/4}}, \tag{11}$$

where we put $a_\nu=\mathrm{Re}(K_\parallel/K_\perp)$, $b_\nu=\mathrm{Im}(K_\parallel/K_\perp)$. The potential is maximum on what is called the resonance cone defined by the equation $z^2+\rho^2a_\nu=0$, if it exists.

The resonance cone *exists* only if a_ν is *negative*. Taking $\alpha=\omega/\omega_{ce}$, $\beta=\omega_{pe}/\omega_{ce}$, and $\gamma=\nu/\omega_{ce}$, this condition can be written as $a_\nu(\alpha,\beta,\gamma)<0$, with

$$\begin{aligned}a_\nu(\alpha,\beta,\gamma)=[&\alpha^8+\alpha^6(-2\beta^2+3\gamma^2-2)\\&+\alpha^4(\beta^4+3\gamma^4-4\beta^2\gamma^2+3\beta^2+1)\\&+\alpha^2(\gamma^6+2\beta^4\gamma^2-2\beta^2\gamma^4-\beta^4+2\gamma^4\\&-\beta^2\gamma^2-\beta^2+\gamma^2)+\beta^4\gamma^2(\gamma^2+1)]/\\&(\alpha^4+\gamma^4+2\alpha^2\gamma^2-2\alpha^2+2\gamma^2+1).\end{aligned} \tag{12}$$

Figure 4 shows the γ dependence of the domains—in terms of $(\omega,\omega_{pe},\omega_{ce},\nu)$—where the resonance cone exists. We see that this domain is reduced by *collisions*: the losses due to electron collisions with the other particles are too important to permit the building of the resonance cone. The upper hybrid branch disappears for $\gamma>0.2209\ldots$; the lower branch for $\gamma>0.4474\ldots$.

For a collisional plasma (the collision frequency ν is nonzero), $K_\parallel$ and $K_\perp$ are complex numbers *a priori*. The potential maximum is obtained for

$$\frac{\rho}{z}=\sqrt{\frac{-1}{a_\nu+(b_\nu^2/a_\nu)}},$$

with $a_\nu<0$, and is written

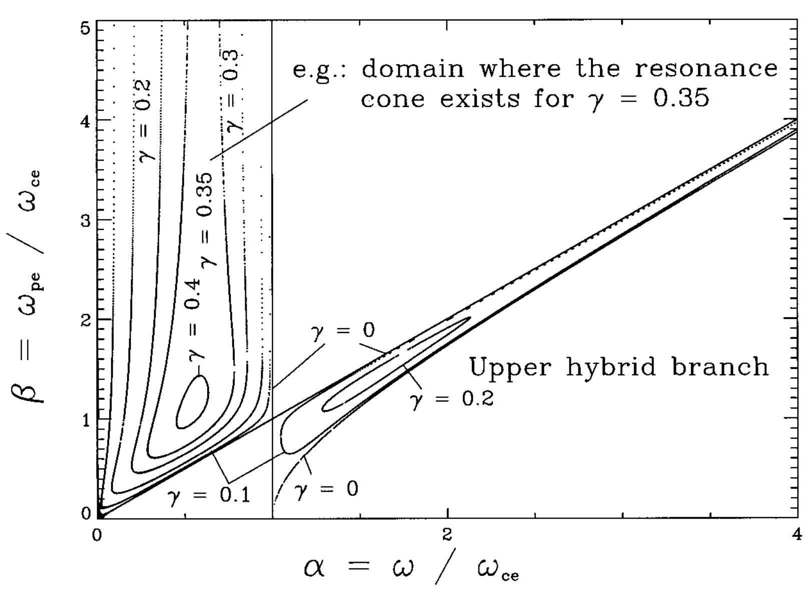

FIG. 4. Domains where the resonance cone exists: $\beta=\omega_{pe}/\omega_{ce}$versus $\alpha=\omega/\omega_{ce}$, depending on $\gamma=\nu/\omega_{ce}$.

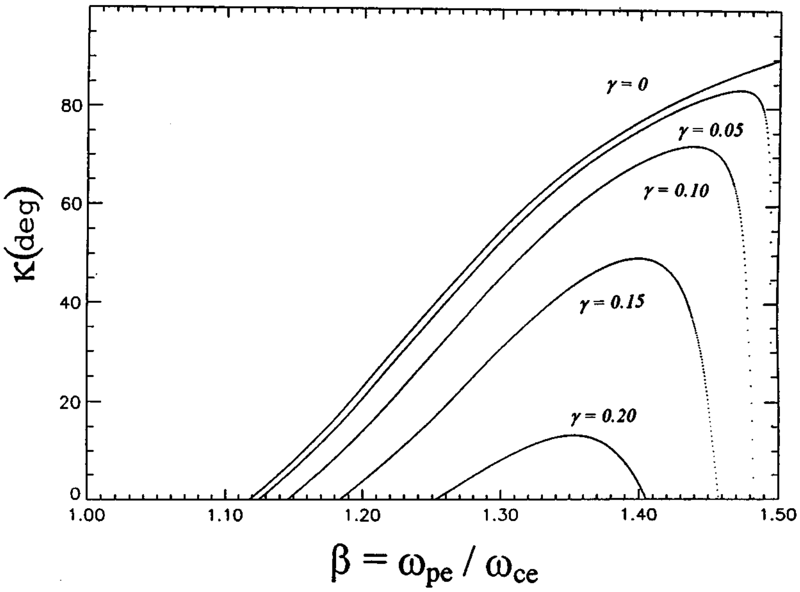

FIG. 5. The resonance cone angle κ (in degrees) versus $\beta=\omega_{pe}/\omega_{ce}$, depending on $\gamma=\nu/\omega_{ce}$, with $\alpha=\omega/\omega_{ce}=1.5$ (upper hybrid branch).

$$\varphi_{\max}(z,t)=\frac{q_e}{8\pi\varepsilon_0}\frac{\exp(-i\omega t)}{K_\perp|z|}\left(\frac{a_\nu^2+b_\nu^2}{b_\nu^2}\right)^{1/4}. \qquad (13)$$

The potential maximum is finite, except for the origin ($z=0$).

For a collisionless plasma ($\nu=0$), the potential maximum is obtained for $\rho/z=\sqrt{-1/a_0}$ with $a_0<0$, and the potential diverges on the resonance cone.

The resonance cone peak is shifted by collisions as we can see in Fig. 5: the cone angle κ decreases when the collision frequency increases. Therefore, the formula (1), used to determine the electronic density n_e, is no longer exact because the cone angle κ depends on the collision frequency. The only thing we can do is to detect the existence of the resonance cone on the upper hybrid branch, and then, in the typical case where $\omega_{pe}\gg\omega_{ce}$, to approximate the plasma electronic pulsation ω_{pe} with the antenna pulsation ω, and then to compute the electronic density.

In conclusion, the frequency domain—in terms of $(\omega,\omega_{pe},\omega_{ce},\nu)$—where the resonance cone exists could be drastically reduced for a collisional plasma and disappears for ''high'' collision frequency. Collisions reduce the resonance cone peak level and widen the resonance peak. Furthermore, because of collisions, the resonance cone peak is shifted, so that formula (1), frequently used to determine the electronic density, is no longer usable. These phenomena are particularly important for cold and weakly magnetized plasmas, such as ionospheric plasmas.

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